Path Spaces of Higher Inductive Types in Homotopy Type Theory

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Preliminary Remarks

Environmental impact of LICS

- European Participant: 3 tons of CO_2 equivalent emissions.
- Carbon Offsetting tries to neutralise impact by saving it elsewhere.
- Why not make it the default?

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Homotopy Type Theory

- For any type $A : \mathcal{U}$ and $x, y : A$ have a type of *equality* proofs $(x = y) : \mathcal{U}$.
- The family $(x = \_) : A \to \mathcal{U}$ is inductively generated by the reflexivity witness $\mathsf{refl} : (x = x)$.
- We can show statements of the form

$$Q : \Pi(x : A).\, x = y \to \mathcal{U}$$

by giving an instance of $Q(x, \mathsf{refl})$. ("J-rule")

Homotopy Type Theory

- A function $f : A \to B$ is an *equivalence* if there are $g, g' : B \to A$ s. t. $f \circ g = \mathrm{id}_B$ and $g' \circ f = \mathrm{id}_A$.
- The *univalence axiom* states that on types, equality and equivalence coincide.
- Types model *spaces*, equality types model *path spaces* $\rightsquigarrow$ Synthetic way to obtain results in topology.

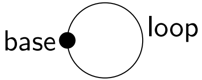
Higher Inductive Types (HITs)

- HITs generalize inductive types to simltaneously generate elements *and* equalities of a type.
- Example: The circle could be written as the followng declaration:

data $\mathbb{S}^1 : \mathcal{U}$ where

base : $\mathbb{S}^1$

loop : base $=$ base



- Specifications for HITs:
  - Via "embedded" type theory (Kaposi & Kovács).
  - In cubical type theory (Coquand, Huber, Mortberg).
  - Based on Homotopy Coequalizers (Lean Theorem Prover).

Homotopy Coequalizers

- A variant of the notion of a *quotient* of a type $A : \mathcal{U}$ by a relation $\sim : A \to A \to \mathcal{U}$.
- $a \sim b$ does not need to be propositional (unique).
- The coequalizer $A/\!\!/\!\sim$ does not need to be a set (unique equalities).
- In pseudo Agda code:

$$
\begin{aligned}
&\text{data } A/\!\!/\!\sim \,:\, \mathcal{U} \text{ where} \\
&\qquad [-] : A \to A/\!\!/\!\sim \\
&\qquad \text{glue} : \Pi\{a, b : A\}.(a \sim b) \to [a] = [b]
\end{aligned}
$$

Examples for Homotopy Coequalizers

- For $A = \mathbf{1}$ and $a \sim b = \mathbf{1}$, $A/\!\!/\!\sim$ is the circle $\mathbb{S}^1$.
- $$\begin{array}{ccc} L & \xrightarrow{\;g\;} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{inr}} \\ M & \xrightarrow[\mathsf{inl}]{} & M \sqcup^L N \end{array}$$
  For types L, M, N, f, g as above consider
  - $A = M + N$ and
  - $\_ \sim \_$ inductively generated by $\mathsf{inl}(f(l)) \sim \mathsf{inr}(g(l))$ for each $l : L$.

  Then, $A/\!\!/\!\sim$ is the *pushout* of M and N along f and g.
- From this get suspensions, sequential colimits, truncation, ...

Encode-decode Proofs

- Common proof strategy when reasoning about HITs
- Examples:
  - To prove $\Omega(\mathbb{S}^1) = ([\star] = [\star]) \simeq \mathbb{Z}$, construct Cover : $\mathbb{S}^1 \to \mathcal{U}$, s. t. $\text{Cover}(x) \simeq ([\star] = x)$ and observe that $\text{Cover}([\star]) \simeq \mathbb{Z}$.
  - Seifert-van Kampen theorem
- In all those cases: Prove the inhabitedness of a family

$$Q : \Pi\{a, b : A\}.([a] = [b]) \to \mathcal{U}$$

The Main Theorem

Theorem

Let $a_0 : A$ *and* $P : \Pi\{b : A\}.[a_0] = [b] \to \mathcal{U}$*. From*

$$r : P(\mathrm{refl}_{[a_0]})$$
$$e : \Pi\{b, c : A\}(q : [a_0] = [b])(s : b \sim c).P(q) \simeq P(q \cdot \mathrm{glue}(s))$$

we can construct

$$\mathrm{ind}_{r,e} : \Pi\{b : A\}(q : [a_0] = [b]).P(q)$$

such that $\mathrm{ind}_{r,e}(\mathrm{refl}) = r$ *and* $\mathrm{ind}_{r,e}(q \cdot \mathrm{glue}(s)) = e(q, s, \mathrm{ind}_{r,e}(q))$*.*

The Non-Dependent Version

Theorem

Let $a_0 : A$ *and* $K : A \to \mathcal{U}$*. For*

$$r : K(a_0)$$
$$e : \Pi\{b, c : A\}.b \sim c \to K(b) \simeq K(c)$$

we have

$$\mathsf{rec}_{r,e} : \Pi\{b : A\}.([a_0] = [b]) \to K(b)$$

with $\mathsf{rec}_{r,e}(\mathsf{refl}_{[a_0]}) = r$ *and* $\mathsf{rec}_{r,e}(q \cdot \mathsf{glue}(s)) = e(s, \mathsf{rec}_{r,e}(q))$ *for* $q : [a_0] = [b]$ *and* $s : b \sim c$*.*

Wild Categories

- Usually, categories in HoTT have *sets*, not types of morphisms.
- Wild categories are not restricted in this way:
  - Objects $|\mathcal{A}| : \mathcal{U}$
  - For $X, Y : |\mathcal{A}|$ have a *type* of morphisms $\mathcal{A}(X, Y) : \mathcal{U}$.
- Most categorical notions are *not* well-behaved.
- Still have *initiality* and *isomorphism* of categories.

The Category of Pointed Families

Let $\mathcal{D}$ be the wild category where objects are pairs (L, p) with

$$L : A /\!\!/ \sim \to \mathcal{U} \text{ and}$$
$$p : L([a_0])$$

and where morphsims in $\mathcal{D}((L, p), (L', p'))$ are pairs (g, ϵ) where

$$g : \Pi(x : A /\!\!/ \sim).L(x) \to L'(x) \text{ and}$$
$$\epsilon : g(p) = p'.$$

Equality induction gives us that $(\lambda x.[a_0] = x, \mathsf{refl})$ is initial in $\mathcal{D}$.

That Other Wild Category

Let $\mathcal{C}$ be the wild category where objects are triples (K, r, e) with

$$K : A \to \mathcal{U},$$
$$r : K(a_0), \text{ and}$$
$$e : \Pi\{b, c : A\}.b \sim c \to K(b) \simeq K(c),$$

and morphisms in $\mathcal{C}((K, r, e), (K', r', e'))$ are triples (f, δ, γ) with

$$f : \Pi(b : A).K(b) \to K'(b),$$
$$\delta : f_{a_0}(r) = r', \text{ and}$$
$$\gamma : \Pi\{b, c : A\}(s : b \sim c).e'(s) \circ f_b = f_c \circ e(s).$$

Both Categories are Isomorphic

Theorem

There is a map $\Phi_0 : |\mathcal{D}| \to |\mathcal{C}|$ *which is an equivalence, as well as a map* $\Phi_1 : \Pi(X, Y : |\mathcal{D}|).\mathcal{D}(X, Y) \to \mathcal{C}(\Phi_0(X), \Phi_0(Y))$ *which is also an equivalence for each* $X, Y : |\mathcal{D}|$*.*

We conclude that $\Phi_0([a_0] = \_, \mathsf{refl})$ is initial in $\mathcal{C}$.

Proof of the Non-Dependent Theorem

- The initial object $\Phi_0([a_0] = \_, \mathsf{refl})$ unfolds to (K^i, p^i, e^i) with

$$K^i(b) = ([a_0] = [b])$$
$$r^i = \mathsf{refl}_{[a_0]}$$
$$e^i = \_ \cdot \mathsf{glue}(s)$$

- The existence of morphisms from (K^i, p^i, e^i) unfolds to the statement of the theorem itself.

Applications

- $\Omega(\mathbb{S}^1) \simeq \mathbb{Z}$ is immediate, given a suitable definition of $\mathbb{Z}$.
- A higher version of Seifert-van Kampen.
- *Embeddings are closed under pushouts.*

Embeddings are Closed Under Pushouts

Definition

A map $f : L \to M$ is called an embedding if

$$\mathsf{ap}_f : \Pi\{l, l' : L\}.\,(l = l') \to (f(l) = f(l'))$$

is a family of equivalences.

Theorem

If f in the diagram on the right is an embedding, so is inr.

$$\begin{array}{ccc} L & \xrightarrow{\;g\;} & N \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{inr}} \\ M & \underset{\mathsf{inl}}{\dashrightarrow} & M \sqcup^{L} N \end{array}$$

Embeddings are Closed Under Pushouts

- To show: The map $\mathsf{ap}_{\mathsf{inr}} : (n_0 = n) \to (\mathsf{inr}(n) = \mathsf{inr}(n_0))$ is an equivalence for all $n, n_0 : N$.
- Fix n_0 and define

$$
\begin{aligned}
Q &: \Pi(m : M + N).(\mathsf{inr}(n_0) = m) \to \mathcal{U} \\
Q(\mathsf{inr}(n), q) &:\equiv \mathsf{ap}_{\mathsf{inr}}^{-1}(q) \\
Q(\mathsf{inl}(m), q) &:\equiv \Sigma((l_0, q_0) : f^{-1}(m)). \\
&\qquad \mathsf{ap}_{\mathsf{inr}}^{-1}\big(g(l_0), q \cdot \mathsf{ap}_{\mathsf{inl}}(q_0) \cdot \mathsf{glue}(l_0)\big)
\end{aligned}
$$

- Our theorem gives us

$$
\mathsf{ind}_{r,e}^{Q} : \Pi\{n : N\}.\,(q : \mathsf{inr}(n_0) = \mathsf{inr}(n)).\mathsf{ap}_{\mathsf{inr}}^{-1}(q)
$$

Conclusions

- We have shown a theorem, similar to an induction principle, to show statements about homotopy coequalizers.
- The theorem can serve as a replacement for encode/decode proofs.

Thank you for your attention!