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Visualization of Hyperbolic Tessellations

Jakob von Raumer | April 9, 2013

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Escher's Art

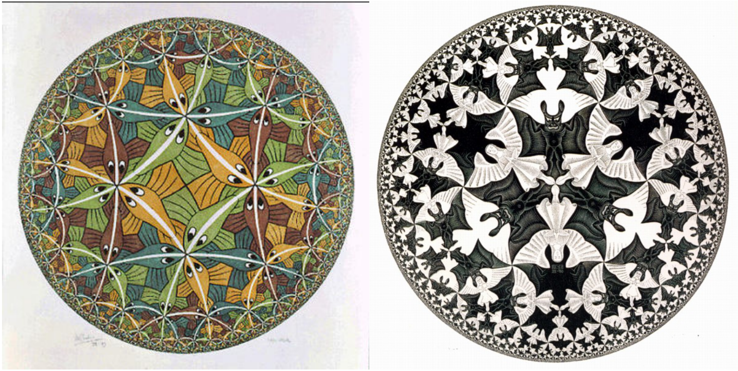

Figure: "Circle Limit III" and "Circle Limit IV" by M. C. Escher, 1959 and 1960

Escher's Art

- Escher's works were inspired by illustrations in a book by H.S.M. Coxeter
- He used woodcuts to replicate the tiles
- To his son George:

> *I had an enthusiastic letter from Coxeter about my colored fish, which I sent him. Three pages of explanation of what I actually did ... . It's a pity that I understand nothing, absolutely nothing of it.*

Aims of my thesis

- Document and summarize theoretical basics of hyperbolic tessellations
- Construct suitable tiles
- Implement algorithms to replicate those

Comparison: Euclidean and hyperbolic Geometry

Euclidean

- For a line g there is *exactly one* line parrallel to g that contain a point $p \notin g$.
- Each triangle has an angle sum of π.

Hyperbolic

- For a line g there are *more than one* lines parallel to g that contain a point $p \notin g$.
- Each triangle has an angle sum of $< \pi$.

Comparison: Euclidean and hyperbolic Geometry

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Length of a path $\gamma : [0,1] \to \mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$:

Euclidean

$$L(\gamma) = \int_0^1 |\gamma'(t)| \,\mathrm{d}t$$

Hyperbolic

$$L(\gamma) = \int_0^1 \frac{|\gamma'(t)|}{\Im(\gamma(t))} \,\mathrm{d}t$$

The distance of two points $a, b \in \mathbb{H}$ is the length of the shortest path between them.

Euclidean

$$d(a,b) = |b - a|$$

Hyperbolic

$$d(a,b) = \ln \frac{|a - \bar{b}| + |a - b|}{|a - \bar{b}| - |a - b|}$$

Geodesics on the upper half-plane

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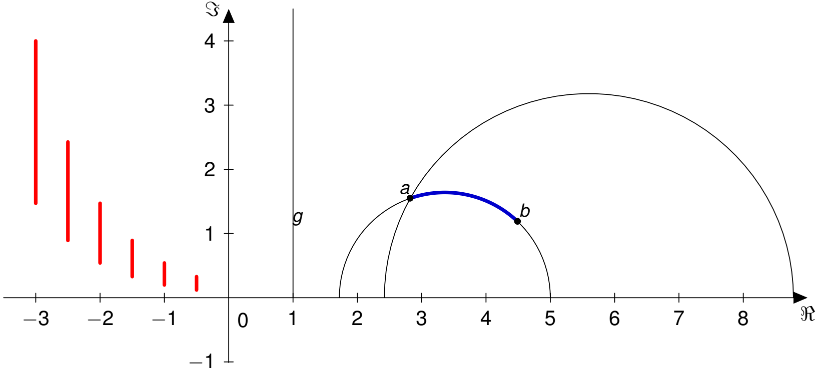


Figure: Geodesics on the upper half-plane.

From the upper half-plane to the Poincaré disk model

The contiuous map $f : \mathbb{H} \to \mathbb{U} : z \mapsto \frac{zi+1}{z+i}$ induces a bounded presentation of $\mathbb{H}$.

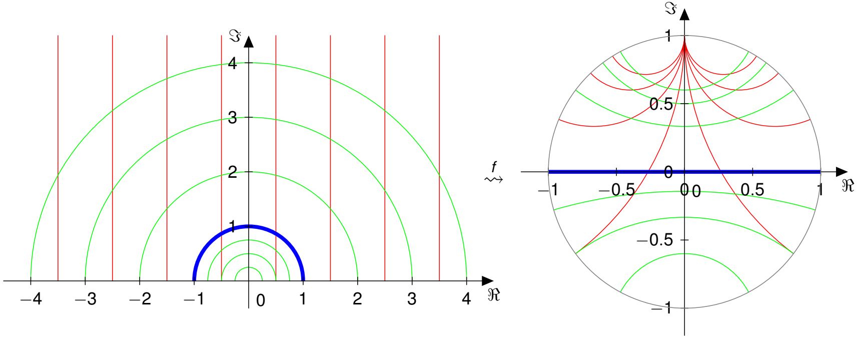


Figure: Geodesics on the upper half-plane and their correspondents on the Poincaré disk model.

The Beltrami-Klein model

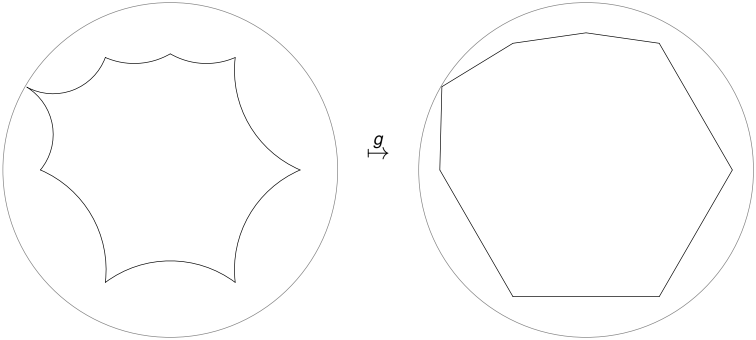


Figure: A polygon, shown in the Poincaré disk model and in the Beltrami-Klein model

Isometries on $\mathbb{H}$

Theorem

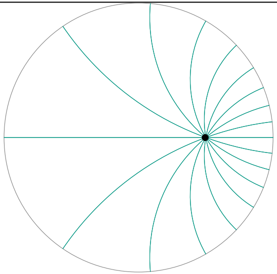
The isometries on $\mathbb{H}$ are a group isomorphic to $PS^*L(2,\mathbb{R}) := S^*L(2,\mathbb{R})/\{\pm I_2\}$.
The orientation preserving isometries on $\mathbb{H}$ are isomorphic to $PSL(2,\mathbb{R}) := SL(2,\mathbb{R})/\{\pm I_2\}$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$ corresponds to the *Möbius transformation* $z \mapsto \frac{az+b}{cz+d}$

Examples:

- Translation $z \mapsto z + 1$
- Dilation $z \mapsto 2z$
- Rotation $z \mapsto -\frac{1}{z}$

3 types of transformations

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is	trace	fixed points	figure
elliptic	$\lvert a+d \rvert < 2$	one in $\mathbb{H}$	
parabolic	$\lvert a+d \rvert = 2$	one at infinity	
hyperbolic	$\lvert a+d \rvert > 2$	two at infinity	

3 types of transformations

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is	trace	fixed points	figure
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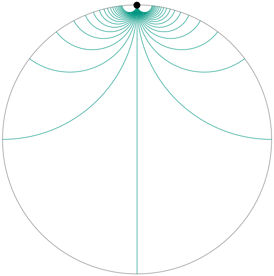
parabolic	$\|a+d\| = 2$	one at infinity	
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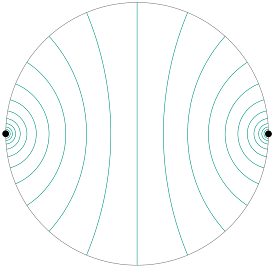
hyperbolic	$\|a+d\| > 2$	two at infinity	

Fuchsian and Kleinian groups

- A discrete subgroup Γ of $PS^*L(2, \mathbb{R})$, is called *Kleinian group*.
- If additionally $\Gamma \leq PSL(2, \mathbb{R})$, then it's called Γ *Fuchsian group*.

Fundamental domains

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Definition

A closed subset $F \subseteq \mathbb{H}$ is a *fundamental domain* for Γ, iff:

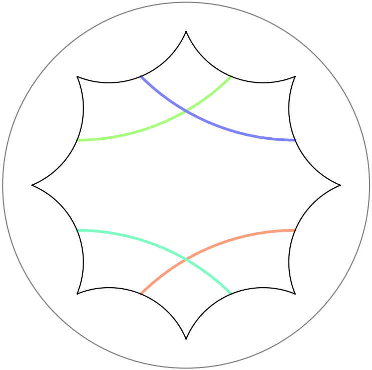
- $\Gamma \cdot F := \bigcup_{T \in \Gamma} T(F) = \mathbb{H}$.
- For all $T \in \Gamma$, F and $T(F)$ intersect only in their boundary.

If F is a fundamental domain for Γ, then $\{T(F) \mid T \in \Gamma\}$ is called a *tessellation*.

Fuchsian Groups: Elliptic and parabolic subgroups

- *Elliptic subgroup*: $\langle T \rangle \leq \Gamma$ where T is elliptic
- *Parabolic subgroup*: $\langle T \rangle \leq \Gamma$ where T is parabolic
- Maximal elliptic or parabolic subgroups which are conjugate to each other, have the same order. They are called *periods* of Γ

The orbit space of a Fuchsian group



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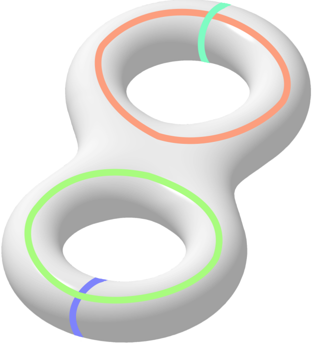


The signature of a Fuchsian group

Definition

Let Γ be a Fuchsian group with periods $m_1, \ldots, m_n \in \mathbb{N}_0 \cup \{\infty\}$, $m_1 \leq \ldots \leq m_n$ and genus $g \in \mathbb{N}_0$. Then the vector $(g, m_1, \ldots, m_n)$ is called the *signature* of Γ.

The program's features

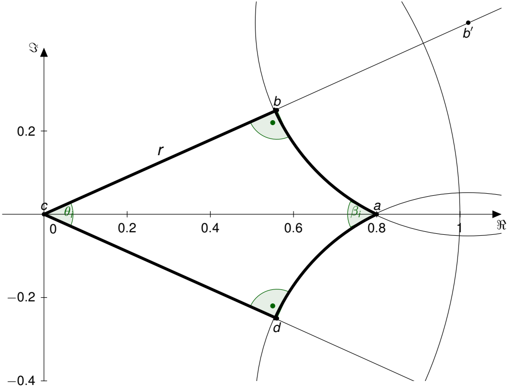
The program should create tessellations which

- are induced by a Fuchsian group with a given signature or
- consists of polygons with a given sequence of inner angles $\frac{2\pi}{m_i}$.

Polygons for tiling by reflections

- $\beta_i = \frac{2\pi}{m_i}$
- $\lim_{r \to 0} \theta_i = \pi$
- $\lim_{r \to 1} \theta_i = 0$

$\Rightarrow$ Find $r_0 \in (0, 1)$ such that $\sum_{i+0}^{n} \theta_i = 2\pi$.



Polygons for tiling by Fuchsian groups

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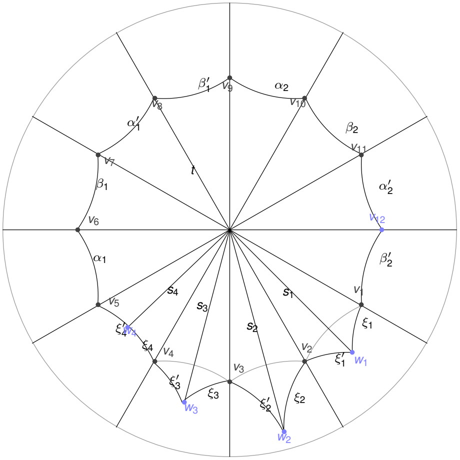


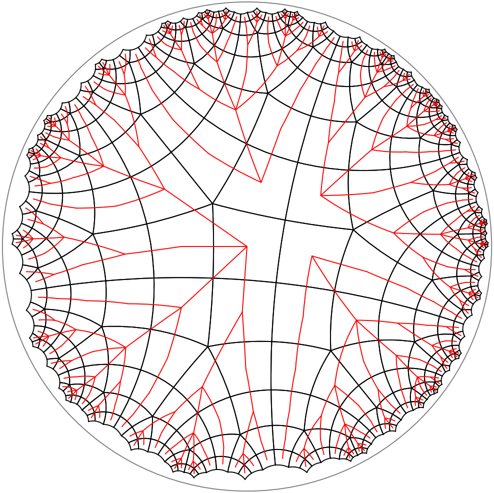
Replication algorithm by Dunham

- It's based on a depth-first search.
- It's a "combinatorial" algorithm: Approach only depends on the corner valencies of the polygon.

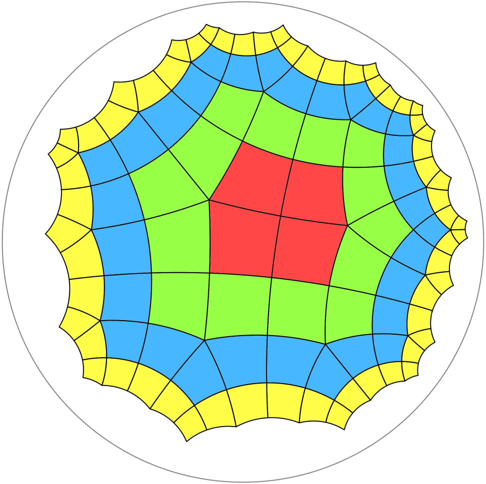
It can replicate arbitrary polygons, except for:

- Triangular fundamental domains or
- at least one corner valency of three.

Search trees of the Dunham algorithm



Separating the tessellation into layers



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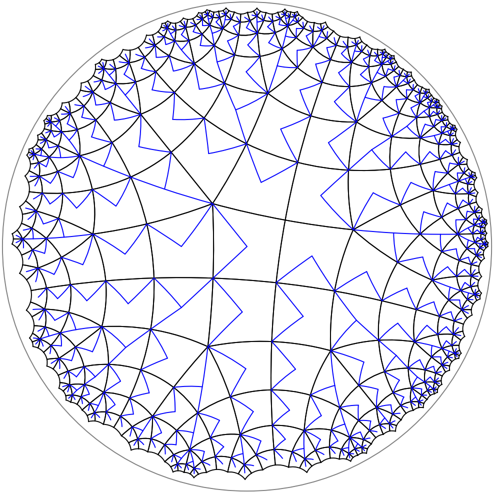


Repliction using a priority queue

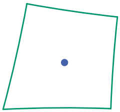
Basic approach:

- Transform each tile by all transformations mapping it to edge adjacent tiles
- Discard tiles we already met

Repliction using a priority queue

Three data structures:

- Liste `inactivePolys`: Polygons yet expanded
- Priority queue `activePolys`: Polygons still to be expanded
- Hash set `midpoints`: Midpoints of polygons we met so far



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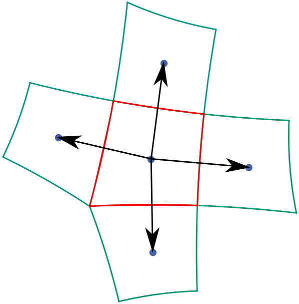

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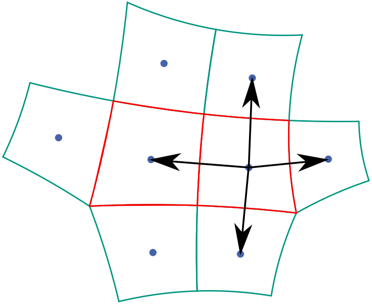
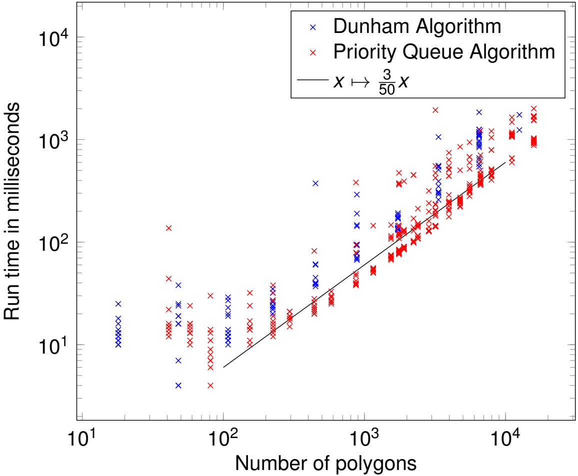


Comparing the run time



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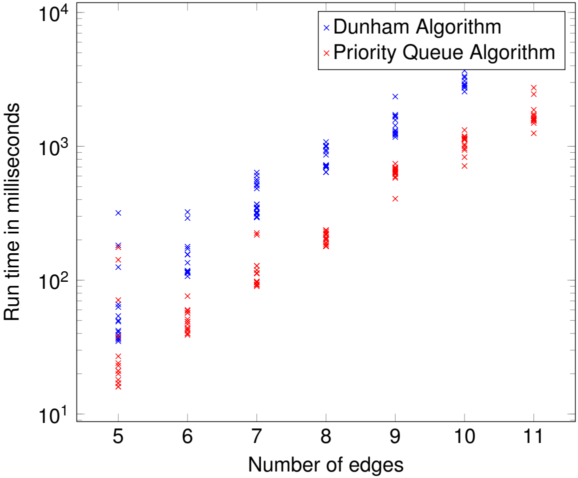

Outlook

Questions and Work still to accomplish:

- Try to solve the restrictions of the Dunham algorithm
- Optimizing the approximation used in the creation of the base polygons
- Replicate arbitrary vector graphics on the base polygons
- Interactive zoom and pan
- Adapting the stroke width according to hyperbolic geometry

On something completely unrelated...

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