Coherence via Well-Foundedness

– Taming Set-Quotients in Homotopy Type Theory

35th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)

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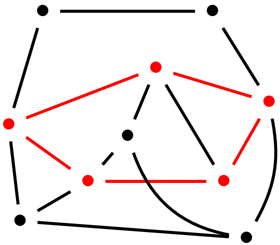
Outline of the Talk

A Graph Theoretic Problem

Noetherian Cycle Induction

Application: Free Groups

General Problem



Consider paths in a graph.

If we want to prove a property. . .

- *for all paths:* **Induction!**
- *for all closed paths:* **how???**

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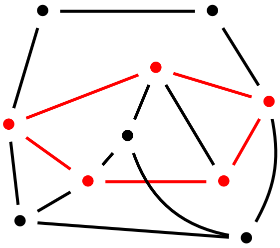

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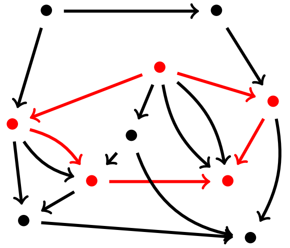
Topic of this talk:

- Approach for a special case of this problem
- Applications in homotopy type theory (HoTT)

Graph Theoretic Formulation



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Assumptions: The graph is

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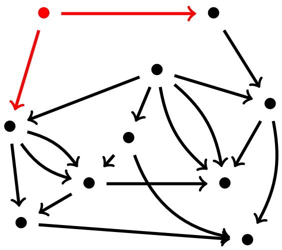

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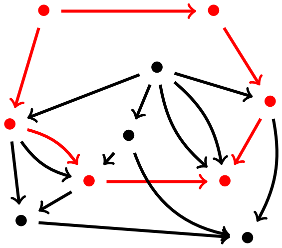

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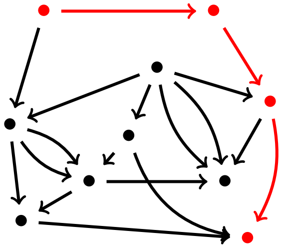

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Assumptions: The graph is

- locally confluent, and
- Noetherian (co-wellfounded).

Example: Reductions in Free Groups

Reduction steps on words in a free group on a set M form such a graph.

$$[m_0^{-1}, m_1, m_1^{-1}, m_2^{-1}, m_3^{-1}, m_3]$$

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Graph Theoretic Formulation

Our proposed solution consists of the following four steps:

1. Given a relation $\rightsquigarrow$ on a set A, we define a new relation $\rightsquigarrow^{\circ}$ on cycles on A.
2. If $\rightsquigarrow$ is Noetherian, then so is $\rightsquigarrow^{\circ}$.
3. If $\rightsquigarrow$ further is locally confluent, then any cycle can be split into a $\rightsquigarrow^{\circ}$-smaller cycle and a confluence cycle
4. Consequence: We can show a property *for all cycles* inductively by showing it *for empty cycles, confluence cycles, and merged cycles*.

Step 1: List Extension

Definition

The *list extension* of a relation $\rightsquigarrow$ on A is a relation $\rightsquigarrow^L$ on $\mathsf{List}(A)$ generated by

$$[\vec{a_1}, a, \vec{a_2}] \rightsquigarrow^L [\vec{a_1}, x_0, x_1, \ldots, x_k, \vec{a_2}]$$

where all x_i are such that $a \rightsquigarrow x_i$.

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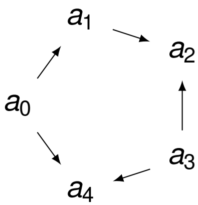
Lemma

If $\rightsquigarrow$ is Noetherian, so is $\rightsquigarrow^L$.

This is similar to the well-founded *multiset extension* by Tobias Nipkow.

Step 2: A Relation on Cycles



Definition

For γ a cycle, write $\varphi(\gamma)$ for the *vertex sequence* of γ.
Write $\gamma \rightsquigarrow^{\circ} \delta$ if $\varphi(\gamma) \rightsquigarrow^{L} \varphi(\delta')$ for any rotation δ' of δ.

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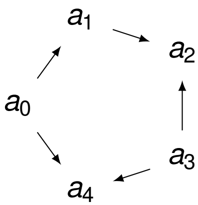

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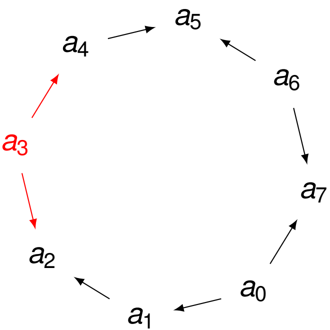
Lemma

If $\rightsquigarrow$ is Noetherian, so is $\rightsquigarrow^\circ$ (and thus also $\rightsquigarrow^{+\circ+}$).

Step 3: Dissecting Cycles



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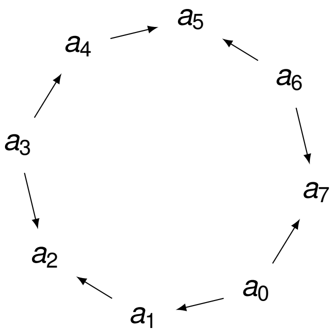

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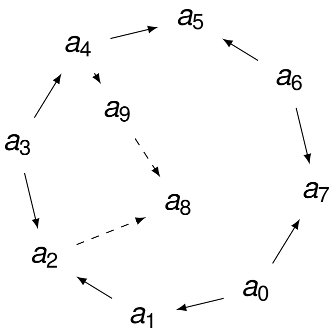

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Then,** $P(\gamma)$ **holds for any cycle** γ**.

Step 4 with Type Theory Flavour

Theorem (Noetherian Cycle Induction)

Given a type A : Type and a Noetherian and locally confluent relation $\rightsquigarrow : A \to A \to \text{Type}$.

Let P : (cycles of $\rightsquigarrow$) $\to$ Type be such that

- *P is stable under rotating of cycles: $P(\alpha\gamma) \to P(\gamma\alpha)$,*
- *P is stable under "merging" of cycles: $P(\alpha\gamma) \to P(\gamma^{-1}\beta) \to P(\alpha\beta)$,*
- *P holds for the empty cycle, and*
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Then, $P(\gamma)$ holds for any cycle γ.

Two Ways to Define Free Groups

How to define the carrier of the free group on a set M?

1. As a (set-)quotient of words $\mathrm{List}(M + M)/\rightsquigarrow$ where the $\rightsquigarrow$ is generated by

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2. As the loop space $\Omega(H, \star)$ of the higher inductive type

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Open question: Do these coincide?

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Approximation: Do their 1-truncations coincide?
Or: Is the fundamental group of the free group trivial?

A Map Between the Definitions

Want to construct: a map $\mathrm{List}(M + M)/\rightsquigarrow \;\to \|\Omega(H, \star)\|_1$

Lemma

Maps $\mathrm{List}(M + M)/\rightsquigarrow \;\to \|\Omega(H, \star)\|_1$ *are equivalently given by triples* (f, h, c) *where*

$$\begin{aligned} f &: \mathrm{List}(M + M) \to \|\Omega(H, \star)\|_1, \\ h &: (\ell_1 \rightsquigarrow \ell_2) \to f(\ell_1) = f(\ell_2), \\ c &: h(\gamma) = \mathrm{refl} \text{ for every cycle } \gamma. \end{aligned}$$

Just set $P(\gamma) :\equiv (h(\gamma) = \mathrm{refl})$ and use cycle induction!

Conclusions

- We found a way to tackle proofs about cycles
- We used it to solve approximations to open problems
- The contents formalised in the Lean theorem prover ($\sim$ 1600 LoC)
- We are exploring applicability
  - to other open problems in HoTT
  - to the field of higher-dimensional rewriting (Thanks to Vincent van Oostrom for his remarks!)
- The speaker of this talk is up for hire!